Int J Theor Phys (2010) 49: 1029–1043
DOI 10.1007/s10773-010-0282-9

Cooper Pair Formation by Quantizing Brownian Motion

Hajime Isimori

Received: 2 November 2009 / Accepted: 16 February 2010 / Published online: 24 February 2010

Abstract We derive a new quantum interaction by quantizing the Brownian motion based on the Nelson equations. By applying the canonical quantization for the equations, interaction as the connection of Brownian and quantum motions appears. Interesting aspect is that it can overcome the Coulomb repulsion if the diffusion coefficient is large enough. As the parameters are mass, diffusion coefficient, and probability density, we can calculate and predict the Cooper pair formation with measurable variables.

Keywords Nelson equations · Canonical quantization · Cooper-pair formation

1 Introduction

We introduce a new phenomenological description of quantum Brownian motion. There are rich phenomena among quantum Brownian motion since it is anticipated to obtain the unified framework of quantum statistical processes such as decoherence, particle creation, dissipation, noise, fluctuation, and diffusion [1–3]. Considering some sort of noise [4–7], the paradigm of statistical mechanics is applicable on quantum physics. There are several methods to represent the quantum Brownian motion often subject to stochastic quantization. Among others, we focus on the stochastic theory of Nelson [8] in the reason of predictivity dealing with all the measurable variables. By projecting conventional quantum system onto his theory, highly predictive model is present. As it produces an attractive interaction between two fermions, we can predict the Cooper pair formation without a free parameter.

Quantization of stochastic process has a long history with various attempts [9, 10]. The most attractive one is the method of Parisi-Wu [11]. It includes an important issue of modern field theory, i.e. a new approach for the gauge theory that it needs neither any gauge fixing nor Faddeev-Popov ghost. Then, as a benefit, it gains new insights for correct nonpertubative path integral formulation of gauge theories [12]. Consistency with conventional quantum field theory like QED and Yang-Mills field was found in several explicit examples [13, 14].

H. Isimori (✉)
Graduate School of Science and Technology, Niigata University, Niigata 950-2181, Japan
e-mail: silviasx3229@yahoo.co.jp

Moreover, fermionic and bosonic fields can be quantized with the stochastic regularization that preserves the symmetries of the original theory [15]. Thus, the theory is regarded as another important way of quantization in addition to canonical and path integral formulations.

On the other hand, the quantization scheme of Nelson has few benefits. This is formulated within the framework of classical Brownian motion. It is actually elegant in the sense that the Schrödinger equation can be derived on the first principle of stochastic theory with simple assumptions. Whereas, as it just explains the appearance of the famous equation aside from important properties of quantum mechanics, this quantization is not helpful to deeply understand the origin of quantum theory. To make matters worse, it suffers from the representation of fermionic and Yang-Mill fields. Considering the success of the standard model of particle physics, this problem is crucial. One might think it is useful if the quantum mechanics can be expressed by the classical stochastic process. Nevertheless, this line of thought makes no sense unless it does influence measurements.

The reason why we revisit the Nelson's theory is that it predicts different dynamics from Newtonian mechanics with measurable parameters. This is very useful to deal with the stochastic modification of quantum mechanics. In the same way to quantize the Newtonian equations, the Nelson equations can be quantized with yielding quantized stochastic interaction. The purpose of this paper is to analyze this interaction. In the view of stochastic quantization of Parisi-Wu, the quantized stochastic interaction is characterized by the noise term of the Langevin equation nonetheless it is disappeared after estimating expectation values if it is a white noise. When the theory is generalized with memory kernel and colored noise, some complicated corrections will appear via modified Green function depending on the noise correlation. For the reason of simple and strong predictions, we prefer the Nelson equations.

There is no difficulty to apply the canonical quantization to the Nelson equations instead of his own scheme. The equations describe the probability density of continuous medium so they are quite similar to the Navier-Stokes equations [16]. With the help of Euler-Lagrange formalism of stochastic mechanics [17, 18] or Navier-Stokes equations [19–21], the Hamiltonian of the system can be derived. The Hamiltonian is written by familiar kinetic and potential terms plus the stochastic potential that is defined in this paper. Parameters of the new potential are mass, diffusion coefficient, and probability density. Because mass is constant and the canonical quantization of probability density is clear, the issue is how to quantize the diffusion. If we invent the quantum osmotic velocity with a hypothesis formulation, quantization of the stochastic potential becomes clear. Strictly speaking, it must be given by formulating quantum diffusion [22–24], but here, we naively quantize them. For the sake of the canonical quantization, fermionic fields are simply represented. After this procedure, we can discuss the quantized stochastic interaction. Interestingly, the interaction becomes attractive between opposite spins.

We apply the stochastic potential to the attractive force between a Cooper-pair. In the BCS theory, the parameter of the force to construct the Cooper-pair remains free. If the stochastic potential plays this role, it can predict the dependence of the diffusion coefficient for the critical temperature and gap equation. Relating to the diffusivity of the wave function of free electrons, Cooper-pairs will appear. With simple calculations, we exhibit the condition for the formulation.

The paper is organized as follows. In Sect. 2, we review the stochastic mechanics of the Nelson's theory and add some modifications for convenience. Section 3 invents a new system of quantum Brownian motion with deriving the stochastic potential that yields quantized stochastic interaction. In Sect. 4, we calculate the perturbation of the potential for hydrogen and helium. In Sect. 5, we analyze the new interaction for two Gaussian wave packets as an approximate eigenstate of two fermions to find new attractive interaction between them. Section 6 applies the stochastic potential as the attraction between two electrons and the condition of Cooper-pair formation is derived. Section 7 devotes the conclusion of our discussion.

2 Stochastic Equations of Motion

Likewise to quantum mechanics, stochastic mechanics compute the probability density ρ [25]. Giving a probability space (Ω, F, P) with state X and a set T of time, where F is a collection $\{F_t : t \in T\}$, F_t is an X-valued random variable, and P is the probability measure, we have a stochastic process $\{X_t : t \in T\}$ defined as a collection of random variables. As for the Markov process it can be represented by a parameter X with respect to $\{F_t\}$. Conventionally, the process can be equally represented by "forward" and "backward" equations. When the probability $u(k,t) = P(X_t = k)$ satisfies forward equations, they allow us to compute all the numbers $u(k, t+1)$ once all the numbers $u(j,t)$ are known. It moves toward from time t to time $t+1$. Similarly, backward equations allow us to move time t to $t-1$. Generally, each evolutions of t to $t+1$ and t to $t-1$ obeys different dynamics. Hence, two types of velocities are required: we define the mean forward velocity $\boldsymbol{b}_+$ and the mean backward velocity $\boldsymbol{b}_-$ for a process $\boldsymbol{x}(t)$ by

$$\begin{aligned} \boldsymbol{b}_+ &\equiv D_+\boldsymbol{x}(t) = \lim_{\epsilon\to+0} \mathcal{E}\left(\frac{\boldsymbol{x}(t+\epsilon)-\boldsymbol{x}(t)}{\epsilon}\right), \\ \boldsymbol{b}_- &\equiv D_-\boldsymbol{x}(t) = \lim_{\epsilon\to-0} \mathcal{E}\left(\frac{\boldsymbol{x}(t+\epsilon)-\boldsymbol{x}(t)}{\epsilon}\right), \end{aligned} \tag{1}$$

where $\mathcal{E}(X)$ is the expectation value at the physical state X.

We assume that the probability density satisfies the Chapman-Kolmogorov equations that dictate dynamics of a particle with $\boldsymbol{b}_+$, $\boldsymbol{b}_-$ and ρ. The Fokker-Planck equations can be derived by up to next leading order of the Chapman-Kolmogorov equations:

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\boldsymbol{b}_+) + \frac{1}{2}D\nabla^2\rho, \qquad \frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\boldsymbol{b}_-) - \frac{1}{2}D\nabla^2\rho, \tag{2}$$

where D is the diffusion coefficient.

Next, we introduce the acceleration of stochastic process, namely mean acceleration. Definition of the mean acceleration is model-dependent [8, 26] and here we also introduce new one.[1] We first define following derivatives

$$\mathcal{D}_\pm \equiv \frac{\partial}{\partial t} + \frac{\boldsymbol{b}_+ + \boldsymbol{b}_-}{2}\cdot\nabla \pm \frac{1}{2}D\nabla^2, \tag{3}$$

---

[1] We define operators $\mathcal{D}_\pm$ with the infinitesimal change: $\tilde{d}\boldsymbol{x}(t) = \boldsymbol{v}(\boldsymbol{x}(t),t)dt + d\boldsymbol{w}(t)$ where $\boldsymbol{w}$ is the Wiener process. A function can be expanded as

$$\tilde{d}f(\boldsymbol{x}(t),t) = \frac{\partial f}{\partial t}(\boldsymbol{x}(t),t)\tilde{d}t + \tilde{d}\boldsymbol{x}(t)\cdot\nabla f(\boldsymbol{x}(t),t) + \frac{1}{2}\sum_{i,j}\tilde{d}x_i(t)\tilde{d}x_j(t)\frac{\partial^2 f}{\partial x_i\partial x_j}(\boldsymbol{x}(t),t).$$

In the same scenario with $\tilde{d}t = dt$, it leads $\mathcal{D}_\pm = D_\pm \mp \boldsymbol{u}\cdot\nabla$. We reinterpret the Langevin equation $\tilde{d}\boldsymbol{v}(t) = -\beta\boldsymbol{v}(t)dt + \boldsymbol{K}(\boldsymbol{x}(t))dt + d\boldsymbol{B}(t)$, then one finds $\mathcal{D}_\pm\boldsymbol{v}(t) = \mp\beta\boldsymbol{v}(t) + \boldsymbol{K}(\boldsymbol{x}(t))$ (see (4b), (8), (9) of [8]). This can lead the mean acceleration of $\boldsymbol{a} = \frac{1}{2}\mathcal{D}_-\boldsymbol{b}_+ + \frac{1}{2}\mathcal{D}_+\boldsymbol{b}_-$. In the limit of differentiable process, $d\boldsymbol{v}$ (under the definition of [8]) corresponds to above one $\tilde{d}\boldsymbol{v} = d\boldsymbol{v}$.

or $D_\pm \mp \frac{\boldsymbol{b}_+ - \boldsymbol{b}_-}{2} \cdot \nabla$. Using them, we define the mean acceleration as

$$\boldsymbol{a} \equiv \frac{1}{2}\mathcal{D}_-\boldsymbol{b}_+ + \frac{1}{2}\mathcal{D}_+\boldsymbol{b}_-. \tag{4}$$

By assuming that a particle performs a Markov process, it will obey the Newtonian equation of motion $\boldsymbol{F} = m\boldsymbol{a}$. The mean acceleration reads

$$\boldsymbol{a} = \frac{1}{2}\frac{\partial}{\partial t}(\boldsymbol{b}_+ + \boldsymbol{b}_-) + \frac{1}{4}((\boldsymbol{b}_+ + \boldsymbol{b}_-)\cdot\nabla)(\boldsymbol{b}_+ + \boldsymbol{b}_-) - \frac{1}{4}D\nabla^2(\boldsymbol{b}_+ - \boldsymbol{b}_-). \tag{5}$$

Comparing to other definitions, a type of solution becomes different.

Now, combining three equations (2), (5), and $\boldsymbol{F} = m\boldsymbol{a}$, Nelson's stochastic equations of motion can be derived:

$$\frac{\partial \boldsymbol{v}}{\partial t} = \frac{\boldsymbol{F}}{m} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \frac{1}{2}D\nabla^2\boldsymbol{u}, \qquad \frac{\partial \boldsymbol{u}}{\partial t} = -\frac{1}{2}D\nabla(\nabla\cdot\boldsymbol{v}) - \nabla(\boldsymbol{u}\cdot\boldsymbol{v}), \tag{6}$$

where $\boldsymbol{u} = (\boldsymbol{b}_+ - \boldsymbol{b}_-)/2$ is the osmotic velocity and $\boldsymbol{v} = (\boldsymbol{b}_+ + \boldsymbol{b}_-)/2$ is the current velocity. To make more convenient equations for our scheme, we rewrite the equations by using the Fokker-Planck equations of next forms

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0, \qquad \boldsymbol{u} - \frac{1}{2}D\nabla\ln\rho = 0. \tag{7}$$

With some calculations, the second equation of (6) reduces the continuity equation. The first equation can be also simplified such that

$$m\frac{\partial \boldsymbol{v}}{\partial t} + m(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \frac{mD^2}{4}\nabla(\nabla^2\ln\rho) = \boldsymbol{F}. \tag{8}$$

Note that the normal distribution can cancel the additional force term.[2] As our motivation is to relate the quantum fluctuation with Brownian motion, above equations of motion are preferred. In the original Nelson equations, normal distribution does not provide constant energy. With this model, interesting phenomena can be found.

New equations of motion can be compared to the Navier-Stokes equations in the sense that the convective derivative of the current velocity $(\partial/\partial t + \boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ and the continuity equation are included. Although it traces the trajectory of a particle, the dynamics is analogous to describe the motion of fluid substance. This relationship can be deeper by applying the mean acceleration to the Navier-Stokes equations: $\rho\boldsymbol{a} = -\nabla p + \nabla\cdot\boldsymbol{T} + \boldsymbol{f}$, where p, $\boldsymbol{T}$ and $\boldsymbol{f}$ are pressure, stress tensor and body force. This assumes pressure and stress tensor can be defined on the probability density. On the other hand, fluid motion might be also described by the modified Navier-Stokes equations. Therefore, there is a possibility to describe the both of particle trajectory and fluid substance by the same equations. This can be viewed as the wave-particle duality though the dynamics is fully classical.

---

[2] In the original equations, the additional force becomes $-\frac{mD^2}{4}\nabla(\nabla^2\ln\rho) - \frac{mD^2}{8}\nabla(\nabla\ln\rho\cdot\nabla\ln\rho)$ which can be rewritten as $-\frac{mD^2}{2}\nabla(\frac{\nabla^2\psi}{\psi})$, where $\psi = e^R$ and $R = \frac{1}{2}\ln\rho$. Obviously, it is close related to the Schrödinger equation.

3 Quantization of the Nelson Equations

We try to quantize the Nelson equations by deriving the Hamiltonian of the system. The Hamiltonian will be derived from the Euler-Lagrange equation. In the context of Nelson's stochastic mechanics, the modified Lagrangian formalism is derived by [17]. Besides, as the equations are quite similar to the Navier-Stokes equations, a technique of Lagrangian formalism of the Navier-Stokes equations is also useful.

Equation (8) can be rewritten as

$$\frac{m}{2}(\mathcal{D}_+ + \mathcal{D}_-)\boldsymbol{v} - \frac{mD^2}{4}\nabla(\nabla^2 \ln\rho) = \boldsymbol{F}. \tag{9}$$

As we have defined the differentiation of the current velocity as this form, the global least action principle will be $\frac{\mathcal{D}_+ + \mathcal{D}_-}{2}\frac{\partial L}{\partial \boldsymbol{v}} - \nabla L = 0$. With this relation, the Lagrangian is derived as[3]

$$L = \frac{1}{2}m\boldsymbol{v}^2 + \frac{mD^2}{4}\nabla^2 \ln\rho - U. \tag{10}$$

Considering the kinetic energy is $m\boldsymbol{v}^2/2$, the Hamiltonian is

$$H = \frac{1}{2}m\boldsymbol{v}^2 - \frac{mD^2}{4}\nabla^2 \ln\rho + U. \tag{11}$$

The second term can be regarded as a new potential which we call stochastic potential. When this potential is constant, the probability density forms the normal distribution.

Now, we try to quantize the Hamiltonian (11). We apply the canonical quantization with the conventional procedure. Then, physical variables are written by eigenvalues of the wave function, e.g., $\hat{\boldsymbol{x}}\langle\boldsymbol{x}|\psi\rangle = \boldsymbol{x}\langle\boldsymbol{x}|\psi\rangle$, $\hat{\boldsymbol{p}}\langle\boldsymbol{x}|\psi\rangle = -i\hbar\nabla\langle\boldsymbol{x}|\psi\rangle$, and $\hat{\rho}(\boldsymbol{x}) = \langle\boldsymbol{x}|\psi\rangle\langle\psi|\boldsymbol{x}\rangle$. Since $\hat{\rho}(\boldsymbol{x})$ is the multi-component function in general, the operation of logarithm has some ambiguity. It can be fixed by considering the quantization of the osmotic velocity $\hat{\boldsymbol{u}}\langle\boldsymbol{x}|\psi\rangle = \boldsymbol{u}\langle\boldsymbol{x}|\psi\rangle$. We assume, as a hypothesis, the logarithm of $\hat{\rho}$ operates for each component of the density matrix rather than operates the matrix itself. As for the quantization of the diffusion coefficient, we assume that it is a constant. For example, supposing the gas kinetic model about defects [22], the diffusion coefficient will be $D \sim Aa^2/\hbar x\sigma$, where A, a, x, and σ are tunneling probability amplitude, lattice constant, fractional concentration, and cross section, respectively. To regard it as a constant, local information should be discarded. Actually, the quantization of the diffusion coefficient will be very complicated, but as we neglect time evolution and take approximate solution, then the assumption will be valid. Hence the osmotic velocity operator can be written by $\hat{\boldsymbol{u}}_{nm} = \frac{1}{2}D\nabla\ln[\hat{\rho}_{nm}(\boldsymbol{x})]$. As the osmotic velocity is measurable, this should be confirmed by experiments.[4]

In this setup, the observable energy $E = \langle\psi|H|\psi\rangle$ can be calculated as

$$E = -\frac{\hbar^2}{2m}\langle\psi|\nabla^2|\psi\rangle - \frac{mD^2}{4}\langle\psi|\nabla^2 \ln[\langle\boldsymbol{x}|\psi\rangle\langle\psi|\boldsymbol{x}\rangle]|\psi\rangle + \langle\psi|U|\psi\rangle. \tag{12}$$

---

[3] It can be also derived from variational principle used for Navier-Stokes equations by taking $L[\boldsymbol{q}_t, \dot{\boldsymbol{q}}_t] = \int_{\Omega_0} d^3a \frac{1}{2}m\dot{\boldsymbol{q}}_t^2(\boldsymbol{a}) + \frac{mD^2}{4}\nabla^2 \ln\rho(\boldsymbol{q}_t(\boldsymbol{a})) - U(\boldsymbol{q}_t(\boldsymbol{a}))$, where $\boldsymbol{q}_t(\boldsymbol{a}) = \boldsymbol{q}(\boldsymbol{a}, t)$ and $\dot{\boldsymbol{q}}_t(\boldsymbol{a}) = \boldsymbol{v}(\boldsymbol{q}(\boldsymbol{a}, t), t)$ [19].

[4] For instance, if the spinor component is mentioned, the quantum osmotic velocity is predicted as $u_i = \frac{1}{2}D\langle\psi_\alpha|\nabla_i \ln\hat{\rho}_{\alpha\beta}(\boldsymbol{x})|\psi_\beta\rangle$ assuming the diffusion coefficient is common to any spinor.

For the plane wave solution $\langle \boldsymbol{x}|\psi\rangle = e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$, the stochastic potential becomes zero. However, linear combination of the solution makes a shift of energy since the system is no more linear. The nonlinear effect becomes important depending on the diffusion coefficient. To recover conventional quantum mechanics, the diffusion coefficient must be suppressed.

In latter sections, we mainly use the Gaussian wave packet because the stochastic potential gains a constant effect:

$$\langle \boldsymbol{x}|\psi\rangle = \prod_{i=1}^{3} \frac{e^{-\frac{(x_i-X_i)^2}{4(\Delta x_i)^2}+i\frac{\boldsymbol{p}\cdot\boldsymbol{x}}{\hbar}}}{(2\pi(\Delta x_i)^2)^{1/4}}, \tag{13}$$

which is normalized as $\int d^3x |\langle \boldsymbol{x}|\psi\rangle|^2 = 1$. The position uncertainty $\Delta\boldsymbol{x}$ is given by the standard deviation

$$\Delta x_i = \sqrt{\langle \psi|x_i^2|\psi\rangle - \langle \psi|x_i|\psi\rangle^2}. \tag{14}$$

Hereafter, we assume $\Delta x_i = \Delta x$ for simplicity. In this case, the energy becomes

$$E = \frac{\boldsymbol{p}^2}{2m} + \frac{\hbar^2}{8m(\Delta x)^2} + \frac{3mD^2}{4(\Delta x)^2} + \langle U\rangle. \tag{15}$$

The stochastic potential consists of mass, diffusion coefficient, and variance of position. Since they are measurable, the stochastic potential can be investigated by experiments. When the measurement systematic error can be reduced for samples of relatively large diffusion coefficient, we expect that the potential can be discovered.

As we use the conventional quantization procedure, generalization of the system to fermionic field, Yang-Mills field, and various interactions of quantum field theory will be possible in the same manner of particle physics. Inclusion of the effect of the Brownian motion can be realized by adding the stochastic potential with definitions of mean acceleration and the quantum osmotic velocity at individual fields.[5] Compared to the Nelson's quantization scheme, our procedure is less elegant but this can overcome several problems, e.g. unnatural assumption $D = \hbar/2m$ that current experiments never support and difficulty of representations of quantum fields except scalar. Further, since our model has strong predictions, it is easier to be proved.

Other notable thing is the correspondence of Brownian and quantum motions. We inherit this heart from the Nelson's theory so that the radical departure between their dynamics does not exist. For both of Brownian and quantum particles, the probability density is described and physical variables must be given by expectation values. As we can interpret that the canonical quantization is merely a constraint of the system, the theory of Brownian motion incorporates quantum mechanics. Although it cannot explain the fundamental scale of the Planck constant, the concept of the dynamics is common. In other words, quantum particles can be regarded as the special type of Brownian particles with the characteristic scale of the Planck constant. On the source of randomness, there seems something special for quantum particles to be characterized by the constant. This consideration is not important here but when considering generalized process such as decoherence, noise, friction, it will make some sense.

---

[5] As photon, weak bosons, and gluons are relativistic particles, Nelson's theory should be extended with relativity. There are some papers dealing with this issue [27, 28].

4 Hydrogen and Helium

In this section, we calculate the energy shift caused by the stochastic potential. With the perturbation using conventional wave functions of hydrogen and helium, the stochastic potential is evaluated by measurable parameters. As the measurement of energy spectrum of simple atoms is easy, this prediction is important.

When the stochastic potential is neglected, the wave function of an electron within a hydrogen is

$$\langle \boldsymbol{x}|\psi\rangle_{n\ell m} = R_{n\ell}(r)Y_{\ell m}(\theta,\varphi), \tag{16}$$

where $R_{n\ell}(r)$ is the radial function and $Y_{\ell m}(\theta,\varphi)$ is the spherical harmonics. We apply usual solutions of the Schrödigner equation to see a perturbative effect. We write the first order of each energy level by $V_{\mathrm{st}}^{(nm\ell)} = -\frac{mD^2}{4}{}_{nm\ell}\langle\psi|\nabla^2\ln\rho(\boldsymbol{x})|\psi\rangle_{nm\ell}$. In the simplest case $\langle \boldsymbol{x}|\psi\rangle_{100} = 2(1/a_0)^{3/2}e^{-r/a_0}/\sqrt{4\pi}$, where a_0 is the Bohr radius, it becomes

$$V_{\mathrm{st}}^{(100)} = -\frac{mD^2}{4}\int d^3x\,\frac{4e^{-\frac{2r}{a_0}}}{4\pi a_0^3}\nabla^2\ln\left[\frac{4e^{-\frac{2r}{a_0}}}{4\pi a_0^3}\right] = \frac{mD^2}{a_0^2}. \tag{17}$$

It is also easy to calculate for $n = 2$, e.g.

$$V_{\mathrm{st}}^{(200)} = -\frac{mD^2}{4}\int d^3x\,\frac{(2-r/a_0)^2e^{-\frac{r}{a_0}}}{4\pi(2a_0)^3}\nabla^2\ln\left[\frac{(2-r/a_0)^2e^{-\frac{r}{a_0}}}{4\pi(2a_0)^3}\right] = \frac{mD^2}{4a_0^2}. \tag{18}$$

In the same way, we can calculate lower energy modes. Some other examples are shown in Table 1.

As a concrete example, let us consider the shift of the Lyman alpha. Generally, the diffusion coefficient will change when an electron is excited, then our prediction is

$$E = -10.2[\mathrm{eV}] + \frac{m(D_1)^2}{a_0^2} - \frac{m(D_2)^2}{4a_0^2}, \tag{19}$$

where D_1 is the diffusion coefficient for $V_{\mathrm{st}}^{(100)}$ and D_2 for $V_{\mathrm{st}}^{(200)}$. If the diffusion coefficient remains the same, red-shift occurs. On the other hand, if atoms have various diffusions, spectrum will be broadened. This can be compared with the Doppler broadening:[6]

$$P(f)df = \sqrt{\frac{mc^2}{2\pi k_B T f_0^2}}\,e^{-\frac{mc^2(f-f_0)^2}{2k_B T f_0^2}}\,df, \tag{20}$$

Table 1 Each value of $V_{\mathrm{st}}^{(n\ell m)}$ in the unit of mD^2/a_0^2. We have not tried the general case, but it might be possible in the case of hydrogen

(ℓ,m)	$(0,0)$	$(1,0)$	$(1,\pm1)$	$(2,0)$	$(2,\pm1)$	$(2,\pm2)$
$n=1$	1					
$n=2$	1/4	1/4	1/8			
$n=3$	1/9	1/9	2/27	1/9	2/27	1/27

---

[6] We refer to wikipedia and it assumes the Maxwell distribution to evaluate the doppler broadening.

where P, f, c, k_B, and T are fraction of particles, frequency, speed of light, Boltzmann constant, and temperature, respectively. Observation of the broadening in the context of the Brownian motion is interesting for the search of new potential.

We also consider the contribution of the stochastic potential for helium. To give a perturbation, we write the Hamiltonian of a helium as $H^{(0)} + H^{(1)}$ with

$$\begin{aligned} H^{(0)} &= -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 - \frac{2e^2}{4\pi\epsilon_0 r_1} - \frac{2e^2}{4\pi\epsilon_0 r_2}, \\ H^{(1)} &= \frac{e^2}{4\pi\epsilon_0|\boldsymbol{r}_1 - \boldsymbol{r}_2|} - \frac{mD^2}{4}(\nabla_1^2 + \nabla_2^2)\ln\hat{\rho}. \end{aligned} \tag{21}$$

The energy of $H^{(0)}$ is $E^{(0)} = -(4\alpha^2 mc^2/2)(1/n_1^2 + 1/n_2^2)$. The next level of the perturbative energy is $\langle\psi_{100}|H^{(1)}|\psi_{100}\rangle = -5mc^2\alpha^2/4 + mD^2/4a_0^2$. The first excited mode is also easy to be calculated. To anti-symmetrize the wave function, we use

$$\begin{aligned} \langle\boldsymbol{x}_1, \boldsymbol{x}_2|\Psi\rangle_+ &= \frac{1}{\sqrt{2}}(\phi_{100}(\boldsymbol{x}_1)\phi_{2\ell m}(\boldsymbol{x}_2) + \phi_{2\ell m}(\boldsymbol{x}_1)\phi_{100}(\boldsymbol{x}_2))\chi_{\text{singlet}}, \\ \langle\boldsymbol{x}_1, \boldsymbol{x}_2|\Psi\rangle_- &= \frac{1}{\sqrt{2}}(\phi_{100}(\boldsymbol{x}_1)\phi_{2\ell m}(\boldsymbol{x}_2) - \phi_{2\ell m}(\boldsymbol{x}_1)\phi_{100}(\boldsymbol{x}_2))\chi_{\text{triplet}}, \end{aligned} \tag{22}$$

where $\chi_{\text{singlet}} = (\chi_+\chi_- - \chi_-\chi_+)/\sqrt{2}$, $\chi_{\text{triplet}} = \{\chi_+\chi_+, (\chi_+\chi_- + \chi_-\chi_+)/\sqrt{2}, \chi_-\chi_-\}$, and $\chi_{+/-}$ is the spin state of up/down. We consider the stochastic potential for the density matrices of $|\Psi\rangle_+ {}_+\langle\Psi|$ and $|\Psi\rangle_- {}_-\langle\Psi|$ as $V_{\text{st}}^{(+)}$ and $V_{\text{st}}^{(-)}$. They can be written by

$$\begin{aligned} V_{\text{st}}^{(\pm)} = -\frac{mD^2}{8}\int & d^3x_1 d^3x_2 (\phi_{100}(\boldsymbol{x}_1)\phi_{2\ell m}(\boldsymbol{x}_2) \pm \phi_{2\ell m}(\boldsymbol{x}_1)\phi_{100}(\boldsymbol{x}_2))^2 \\ & \times (\nabla_1^2 + \nabla_2^2)\ln(\phi_{100}(\boldsymbol{x}_1)\phi_{2\ell m}(\boldsymbol{x}_2) \pm \phi_{2\ell m}(\boldsymbol{x}_1)\phi_{100}(\boldsymbol{x}_2))^2. \end{aligned} \tag{23}$$

When $\ell = m = 0$, the equation yields $V_{\text{st}} = 5mD^2/4a_0^2$. For $\ell = 1$ and $m = 0$, we have the same result $V_{\text{st}} = 5mD^2/4a_0^2$. Unlike them, for $\ell = 1$ and $m = \pm 1$, the calculation is so complicated that we cannot get a precise result. Generally, for the system with many electrons, the nonlinearity will make the calculation more difficult. In addition, as the diffusion coefficient has the dependence of number density, the perturbation will be useless if the diffusion becomes large enough. Though the stochastic potential becomes important by increasing the number of electrons, the theoretical predictions must be hard.

5 Attraction from the Stochastic Potential

We analyze the stochastic potential about two particles of free motions. Since the Schrödigner equation is now nonlinear, the quantum state will be generally complicated. However, it approximately contains a simple solution, that is, two isolated Gaussian distributions. If the overlap is small, the probability density of each particle can be adopted to the Gaussian wave packet individually. As a simple example, let us consider two scalar fields written by $\langle\boldsymbol{x}_1, \boldsymbol{x}_2|\phi_1, \phi_2\rangle = \phi_1(\boldsymbol{x}_1)\phi_2(\boldsymbol{x}_2) + \phi_1(\boldsymbol{x}_2)\phi_2(\boldsymbol{x}_1)$ with assuming $\phi_1(\boldsymbol{x})\phi_2(\boldsymbol{x}) \approx 0$. Using this approximation, the Schroödinger equation reduces to

$$\begin{aligned} &H\phi_1(\boldsymbol{x}_1)\phi_2(\boldsymbol{x}_2) \\ &\quad\approx \left(-\sum_{n=1}^{2}\frac{\hbar^2}{2m_n}\nabla_n^2 - \sum_{n=1}^{2}\frac{m_n D_n^2}{4}\nabla_n^2 \ln[|\phi_1(\boldsymbol{x}_1)|^2\,|\phi_2(\boldsymbol{x}_2)|^2]\right)\phi_1(\boldsymbol{x}_1)\phi_2(\boldsymbol{x}_2). \end{aligned} \tag{24}$$

Thereby, we can apply the normal distribution to each field. Nonlinear analysis will be important when they are very near since a new eigenstate of the nonlinear Schrödinger equation may arise. Further, the configuration of the distributions will be changing as time evolves. Although exotic phenomena might be found by considering the time evolution with the nonlinear Hamiltonian, we treat the two normal distributions as stable enough to neglect any nonlinear solution. Our statement of finding new attractive interaction means for the two isolated Gaussian waves. If the above approximation is not valid, the analysis might be quite different.

In the same way, we apply to fermions with the antisymmetric wave-function

$$|\Psi\rangle = \frac{1}{\sqrt{\rho_0}}(|\psi_1\psi_2\rangle - |\psi_2\psi_1\rangle), \tag{25}$$

where $\rho_0 = (\langle\psi_1\psi_2| - \langle\psi_2\psi_1|)(|\psi_1\psi_2\rangle - |\psi_2\psi_1\rangle)$ is the normalization factor. For each particle, free motion with the Gaussian fluctuation is assumed. We represent the wave function of the spin-$1/2$ by $\langle\boldsymbol{x}|\psi_n\rangle = \phi_n(\boldsymbol{x})\chi_n$ where

$$\phi_1(\boldsymbol{x}) = \frac{e^{-\frac{(\boldsymbol{x}-\boldsymbol{X}_1)^2}{4(\Delta x_1)^2}+i\frac{\boldsymbol{p}_1\cdot\boldsymbol{x}}{\hbar}}}{(\sqrt{2\pi}\,\Delta x_1)^{3/2}}, \qquad \phi_2(\boldsymbol{x}) = \frac{e^{-\frac{(\boldsymbol{x}-\boldsymbol{X}_2)^2}{4(\Delta x_2)^2}+i\frac{\boldsymbol{p}_2\cdot\boldsymbol{x}}{\hbar}}}{(\sqrt{2\pi}\,\Delta x_2)^{3/2}}, \tag{26}$$

and χ_n are arbitrary spin states which are eigenstates of $\boldsymbol{e}_n\cdot\hat{\boldsymbol{S}}$. Two-component spinor can be explicitly given as

$$\begin{aligned} &\chi_n = e^{-\frac{i\varphi_n}{2}}\cos\left(\frac{\theta_n}{2}\right)\begin{pmatrix}1\\0\end{pmatrix} + e^{\frac{i\varphi_n}{2}}\sin\left(\frac{\theta_n}{2}\right)\begin{pmatrix}0\\1\end{pmatrix}, \qquad \hat{\boldsymbol{S}} = \frac{\hbar}{2}\hat{\boldsymbol{\sigma}}, \\ &\boldsymbol{e}_n = \sin\theta_n\cos\varphi_n\boldsymbol{e}_x + \sin\theta_n\sin\varphi_n\boldsymbol{e}_y + \cos\theta_n\boldsymbol{e}_z. \end{aligned} \tag{27}$$

The Gaussian solution can be adopted when $\phi_1(\boldsymbol{x})\phi_2(\boldsymbol{x}) \approx 0$ that is $|\boldsymbol{X}_1 - \boldsymbol{X}_2| \gg \Delta x_1, \Delta x_2$. We can calculate ρ_0 as

$$\rho_0 = 1 - |k_{21}|^2|c_1c_2^* + s_1s_2^*|^2, \tag{28}$$

where $c_n = e^{-i\varphi_n/2}\cos(\theta_n/2)$, $s_n = e^{i\varphi_n/2}\sin(\theta_n/2)$, and $k_{21} = \int d^3x\phi_2^*(\boldsymbol{x})\phi_1(\boldsymbol{x})$ or

$$\begin{aligned} k_{21} &= \left(\sqrt{\frac{2\Delta x_1\Delta x_2}{(\Delta x_1)^2 + (\Delta x_2)^2}}\right)^3 \\ &\quad\times e^{-\frac{(\boldsymbol{X}_1-\boldsymbol{X}_2)^2}{4((\Delta x_1)^2+(\Delta x_2)^2)} - \frac{(\Delta x_1\Delta x_2)^2(\boldsymbol{p}_1-\boldsymbol{p}_2)^2}{(\Delta x_1)^2+(\Delta x_2)^2} + i\frac{(\boldsymbol{p}_1-\boldsymbol{p}_2)\cdot((\Delta x_2)^2\boldsymbol{x}_1+(\Delta x_1)^2\boldsymbol{x}_2)}{(\Delta x_1)^2+(\Delta x_2)^2}}. \end{aligned} \tag{29}$$

With a short hand notation $\chi_{ijk\ell} = \chi_i \chi_j \chi_k^\dagger \chi_\ell^\dagger$, the probability density function becomes

$$\hat{\rho}(\boldsymbol{x}_1, \boldsymbol{x}_2) = -\frac{1}{2(2\pi \Delta x_1 \Delta x_2)^3 \rho_0} \Big(e^{-\frac{(\boldsymbol{x}_1 - \boldsymbol{X}_1)^2}{2(\Delta x_1)^2} - \frac{(\boldsymbol{x}_2 - \boldsymbol{X}_2)^2}{2(\Delta x_2)^2}} \chi_{1212} + e^{-\frac{(\boldsymbol{x}_2 - \boldsymbol{X}_1)^2}{2(\Delta x_1)^2} - \frac{(\boldsymbol{x}_1 - \boldsymbol{X}_2)^2}{2(\Delta x_2)^2}} \chi_{2121} \\ - e^{-\frac{(\boldsymbol{x}_1 - \boldsymbol{X}_1)^2 + (\boldsymbol{x}_2 - \boldsymbol{X}_1)^2}{4(\Delta x_1)^2}} e^{-\frac{(\boldsymbol{x}_1 - \boldsymbol{X}_2)^2 + (\boldsymbol{x}_2 - \boldsymbol{X}_2)^2}{4(\Delta x_2)^2}} \\ \times \Big(e^{2i \frac{(\boldsymbol{p}_1 - \boldsymbol{p}_2) \cdot (\boldsymbol{x}_1 - \boldsymbol{x}_2)}{\hbar}} \chi_{1221} + e^{-2i \frac{(\boldsymbol{p}_1 - \boldsymbol{p}_2) \cdot (\boldsymbol{x}_1 - \boldsymbol{x}_2)}{\hbar}} \chi_{2112} \Big) \Big). \tag{30}$$

5.1 Kinetic Energy

Let us calculate the kinetic energy

$$K = -\frac{1}{2\rho_0} \int d^3x_1 d^3x_2 (\phi_1^*(\boldsymbol{x}_1)\phi_2^*(\boldsymbol{x}_2)\chi_1^\dagger \chi_2^\dagger - \phi_2^*(\boldsymbol{x}_1)\phi_1^*(\boldsymbol{x}_2)\chi_2^\dagger \chi_1^\dagger) \\ \times \left(\frac{\hbar^2}{2m_1} \nabla_1^2 + \frac{\hbar^2}{2m_2} \nabla_2^2 \right) (\phi_1(\boldsymbol{x}_1)\phi_2(\boldsymbol{x}_2)\chi_1\chi_2 - \phi_2(\boldsymbol{x}_1)\phi_1(\boldsymbol{x}_2)\chi_2\chi_1). \tag{31}$$

Writing $\ell_{nm} = \int d^3x \phi_n^*(\boldsymbol{x}) \nabla^2 \phi_m(\boldsymbol{x})$, it becomes

$$K = -\frac{\hbar^2}{2\rho_0} \left(\frac{\ell_{11} + \ell_{22}}{2m_1} + \frac{\ell_{11} + \ell_{22}}{2m_2} - 2\left(\frac{1}{2m_1} + \frac{1}{2m_2} \right) \mathrm{Re}[\ell_{21} k_{12}] |c_1 c_2^* + s_1 s_2^*|^2 \right). \tag{32}$$

For a simple case $\Delta x_1 = \Delta x_2$ and $\boldsymbol{p}_1 = \boldsymbol{p}_2 = 0$, we have

$$K = \frac{3\hbar^2}{4\rho_0 (\Delta x_1)^2} \left(\frac{1}{2m_1} + \frac{1}{2m_2} \right) \\ \times \left(1 - \left(1 - \frac{(\boldsymbol{X}_1 - \boldsymbol{X}_2)^2}{12(\Delta x_1)^2} \right) e^{-\frac{(\boldsymbol{X}_1 - \boldsymbol{X}_2)^2}{4(\Delta x_1)^2}} |c_1 c_2^* + s_1 s_2^*|^2 \right). \tag{33}$$

The kinetic energy gains $|\boldsymbol{X}_2 - \boldsymbol{X}_1|$ dependence when $|c_1 c_2^* + s_1 s_2^*| \neq 0$. Nonetheless, it cannot combine a pair of two fermions due to the factor $1/\rho_0$.

5.2 Stochastic Potential: The Same Spin

The density matrix for the same spin, considering spin up, reads $\hat{\rho}_{11}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \rho(\boldsymbol{x}_1, \boldsymbol{x}_2)$ and empty in other elements. The Stochastic potential can be expressed as

$$V_{\mathrm{st}} = \int d^3x_1 d^3x_2 \left(\frac{m_1 D_1^2}{4} \frac{(\nabla_1 \rho(\boldsymbol{x}_1, \boldsymbol{x}_2))^2}{\rho(\boldsymbol{x}_1, \boldsymbol{x}_2)} + \frac{m_2 D_2^2}{4} \frac{(\nabla_2 \rho(\boldsymbol{x}_1, \boldsymbol{x}_2))^2}{\rho(\boldsymbol{x}_1, \boldsymbol{x}_2)} \right). \tag{34}$$

When $\Delta x_1 = \Delta x_2$, we have

$$\rho(\boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{2e^{-\sum_n \frac{(\boldsymbol{x}_1 - \boldsymbol{X}_n)^2 + (\boldsymbol{x}_2 - \boldsymbol{X}_n)^2}{4(\Delta x_1)^2}}}{(2\pi(\Delta x_1)^2)^3 \rho_0} \\ \times \left(\cosh\left[\frac{(\boldsymbol{x}_1 - \boldsymbol{x}_2) \cdot (\boldsymbol{X}_1 - \boldsymbol{X}_2)}{2(\Delta x_1)^2} \right] - \cos\left[\frac{(\boldsymbol{p}_1 - \boldsymbol{p}_2) \cdot (\boldsymbol{x}_1 - \boldsymbol{x}_2)}{\hbar} \right] \right). \tag{35}$$

To insert this one, a difficult part of the calculation is

$$\begin{aligned}
&\int d^3x_1 d^3x_2 \left(\frac{-\boldsymbol{x}_1+\boldsymbol{X}_1}{2(\Delta x_1)^2}-\frac{-\boldsymbol{x}_1+\boldsymbol{X}_2}{2(\Delta x_2)^2}\right)^2 \frac{(|\phi_1(\boldsymbol{x}_1)|^2|\phi_2(\boldsymbol{x}_2)|^2-|\phi_1(\boldsymbol{x}_2)|^2|\phi_2(\boldsymbol{x}_1)|^2)^2}{4\rho_0^2\,\rho(\boldsymbol{x}_1,\boldsymbol{x}_2)} \\
&\quad=\frac{(\Delta x_1)^3(\pi)^{3/2}e^{-\frac{(\boldsymbol{X}_1-\boldsymbol{X}_2)^2}{4(\Delta x_1)^2}}}{\rho_0(2\pi(\Delta x_1)^2)^3}\left(\frac{\boldsymbol{X}_1-\boldsymbol{X}_2}{2(\Delta x_1)^2}\right)^2 \\
&\quad\quad\times\int d^3x_1 \frac{e^{-\frac{\boldsymbol{x}_1^2}{4(\Delta x_1)^2}}\sinh^2(\frac{(\boldsymbol{X}_1-\boldsymbol{X}_2)\cdot\boldsymbol{x}_1}{2(\Delta x_1)^2})}{\cosh(\frac{(\boldsymbol{X}_1-\boldsymbol{X}_2)\cdot\boldsymbol{x}_1}{2(\Delta x_1)^2})-\cos[\frac{(\boldsymbol{p}_1-\boldsymbol{p}_2)\cdot\boldsymbol{x}_1}{\hbar}]}.
\end{aligned}\tag{36}$$

If $\boldsymbol{p}_2-\boldsymbol{p}_1=0$, we can integrate it as

$$\int d^3x_1 \frac{e^{-\frac{\boldsymbol{x}_1^2}{4(\Delta x_1)^2}}\sinh^2[\frac{(\boldsymbol{X}_1-\boldsymbol{X}_2)\cdot\boldsymbol{x}_1}{2(\Delta x_1)^2}]}{\cosh[\frac{(\boldsymbol{X}_1-\boldsymbol{X}_2)\cdot\boldsymbol{x}_1}{2(\Delta x_1)^2}]-1}=8\pi\sqrt{\pi}(\Delta x_1)^3\big(1+e^{\frac{(\boldsymbol{X}_1-\boldsymbol{X}_2)^2}{4(\Delta x_1)^2}}\big).\tag{37}$$

Using them, we find

$$\begin{aligned}
\int d^3x_1 d^3x_2\frac{(\nabla_1\rho(\boldsymbol{x}_1,\boldsymbol{x}_2))^2}{\rho(\boldsymbol{x}_1,\boldsymbol{x}_2)}&=\frac{(\boldsymbol{X}_1-\boldsymbol{X}_2)^2}{4\rho_0(\Delta x_1)^4}\big(1+e^{-\frac{(\boldsymbol{X}_1-\boldsymbol{X}_2)^2}{4(\Delta x_1)^2}}\big) \\
&\quad+\frac{1}{2\rho_0}\left(\frac{6-6|k_{21}|^2}{(\Delta x_1)^2}+\frac{(\boldsymbol{X}_1-\boldsymbol{X}_2)^2}{2(\Delta x_1)^4}\right)-\frac{(\boldsymbol{X}_1-\boldsymbol{X}_2)^2}{2\rho_0(\Delta x_1)^4}.
\end{aligned}\tag{38}$$

It leads to

$$V_{\mathrm{st}}=\frac{m_1D_1^2+m_2D_2^2}{4}\left(\frac{(\boldsymbol{X}_1-\boldsymbol{X}_2)^2}{4\rho_0(\Delta x_1)^4}e^{-\frac{(\boldsymbol{X}_1-\boldsymbol{X}_2)^2}{4(\Delta x_1)^2}}+\frac{3}{\rho_0(\Delta x_1)^2}(1-|k_{21}|^2)\right).\tag{39}$$

This is always repulsive for any D.

5.3 Stochastic Potential: Spin Up-Down

We consider spin up-down correlation with $c_1=s_2=0$ so that $\rho_0=1$. The stochastic potential can be easily evaluated by calculating

$$\begin{aligned}
&\int d^3x_1 d^3x|\phi_1(\boldsymbol{x}_1)|^2|\phi_2(\boldsymbol{x}_2)|^2\left(-\frac{m_1D_1^2}{4}\nabla_1^2-\frac{m_2D_2^2}{4}\nabla_2^2\right)\ln[|\phi_1(\boldsymbol{x}_1)|^2|\phi_2(\boldsymbol{x}_2)|^2] \\
&\quad=\frac{3m_1D_1^2}{4(\Delta x_1)^2}+\frac{3m_2D_2^2}{4(\Delta x_2)^2},
\end{aligned}\tag{40}$$

and

$$\begin{aligned}
&\int d^3x_1 d^3x_2\ \phi_{12}(\boldsymbol{x}_1)\phi_{12}^*(\boldsymbol{x}_2)\left(-\frac{m_1D_1^2}{4}\nabla_1^2-\frac{m_2D_2^2}{4}\nabla_2^2\right)\ln[\phi_{12}(\boldsymbol{x}_1)\phi_{12}^*(\boldsymbol{x}_2)] \\
&\quad=\frac{3(m_1D_1^2+m_2D_2^2)}{4}\frac{(\Delta x_1)^2+(\Delta x_2)^2}{2(\Delta x_1\Delta x_2)^2}|k_{21}|^2,
\end{aligned}\tag{41}$$

where $\phi_{12}(\boldsymbol{x}) = \phi_1^*(\boldsymbol{x})\phi_2(\boldsymbol{x})$. Then we find

$$V_{\text{st}} = \frac{3(m_1 D_1^2 + m_2 D_2^2)((\Delta x_1)^2 + (\Delta x_2)^2)}{4(\Delta x_1 \Delta x_2)^2}(1 - |k_{21}|^2). \tag{42}$$

This is an attractive potential toward $\boldsymbol{X}_1 = \boldsymbol{X}_2$. It becomes important for the next section with the possibility to make Cooper pairs.

5.4 Coulomb Potential

The Coulomb potential is also considered. We simply calculate $\langle\Psi|\frac{e^2}{4\pi\epsilon_0|\boldsymbol{x}_1-\boldsymbol{x}_2|}|\Psi\rangle$, then the main part is

$$\int d^3x_1 d^3x_2 \frac{e^2|\phi_1(\boldsymbol{x}_1)|^2|\phi_2(\boldsymbol{x}_2)|^2}{4\pi\epsilon_0\rho_0|\boldsymbol{x}_1 - \boldsymbol{x}_2|} = \frac{e^2}{4\pi\epsilon_0\rho_0|\boldsymbol{X}_1 - \boldsymbol{X}_2|}\text{erf}\left[\frac{|\boldsymbol{X}_1 - \boldsymbol{X}_2|}{\sqrt{2(\Delta x_1)^2 + 2(\Delta x_2)^2}}\right], \tag{43}$$

where erf$[x]$ denotes the error function. For the same spin, it includes exchange interaction. In the case $\boldsymbol{p}_1 = \boldsymbol{p}_2$, it can be calculated as

$$-\int d^3x_1 d^3x_2 \frac{e^2\phi_1^\dagger(\boldsymbol{x}_1)\phi_2(\boldsymbol{x}_1)\phi_2^\dagger(\boldsymbol{x}_2)\phi_1(\boldsymbol{x}_2)}{4\pi\epsilon_0|\boldsymbol{x}_1 - \boldsymbol{x}_2|} = \frac{-8e^2(\Delta x_1\Delta x_2)^2 e^{-\frac{(\boldsymbol{X}_1-\boldsymbol{X}_2)^2}{2(\Delta x_1)^2+2(\Delta x_2)^2}}}{4\pi\epsilon_0\rho_0\sqrt{2\pi}((\Delta x_1)^2 + (\Delta x_2)^2)^{5/2}}. \tag{44}$$

We can see that the Coulomb potential with the exchange potential is always repulsive. Thus two electrons with the same spin cannot be attracted in our model.

6 Cooper Pair Formation

In the previous section, we have calculated the energy including the interaction between two Gaussian waves of the approximate solution. With the aid of the stochastic potential, two electrons of opposite spins can be combined if the potential is strong enough. Mainly, it needs two conditions to make a pair, i.e. large diffusion coefficient and small position fluctuation.

Considering two electrons of spins up and down, the total energy is

$$\begin{aligned} E = {} & \frac{\hbar^2}{2m_e}\left(\frac{3}{4}\left(\frac{1}{(\Delta x_1)^2} + \frac{1}{(\Delta x_2)^2}\right) + \frac{\boldsymbol{p}_1^2 + \boldsymbol{p}_2^2}{\hbar^2}\right) \\ & + \frac{3m_e(D_1^2 + D_2^2)((\Delta x_1)^2 + (\Delta x_2)^2)}{4(\Delta x_1\Delta x_2)^2}(1 - |k_{21}|^2) + \frac{e^2\text{erf}[\frac{|\boldsymbol{X}_1-\boldsymbol{X}_2|}{\sqrt{2(\Delta x_1)^2+2(\Delta x_2)^2}}]}{4\pi\epsilon_0|\boldsymbol{X}_1 - \boldsymbol{X}_2|}. \end{aligned} \tag{45}$$

When we consider the conditions of $D_1 = D_2$ and $\Delta x_1 = \Delta x_2$ with denoting D_e and Δx_e, it is simplified as

$$E = \frac{\hbar^2}{2m_e}\left(\frac{3}{2(\Delta x_e)^2} + \frac{\boldsymbol{p}_1^2 + \boldsymbol{p}_2^2}{\hbar^2}\right) + \frac{3m_e D_e^2}{(\Delta x_e)^2}(1 - |k_{21}|^2) + \frac{e^2\text{erf}[\frac{|\boldsymbol{X}_1-\boldsymbol{X}_2|}{2\Delta x_e}]}{4\pi\epsilon_0|\boldsymbol{X}_1 - \boldsymbol{X}_2|}. \tag{46}$$

In order to find the potential minimum, we write $|\boldsymbol{X}_1 - \boldsymbol{X}_2| = r$ and consider differentiation of the energy

$$\frac{\partial E}{\partial r} = \frac{3m_e D_e^2 r e^{-\frac{r^2}{4(\Delta x_e)^2} - \frac{(\Delta x_e)^2(\boldsymbol{p}_1-\boldsymbol{p}_2)^2}{\hbar^2}}}{2(\Delta x_e)^4} + \frac{e^2}{4\pi\epsilon_0}\left(\frac{e^{-\frac{r^2}{4(\Delta x_e)^2}}}{\sqrt{\pi}\Delta x_e r} - \frac{\text{erf}[\frac{r}{2\Delta x_e}]}{r^2}\right). \tag{47}$$

Fig. 1 This line is plotted by solving $\partial E/\partial r = 0$ of (47) and choosing larger r. If k_s is larger than about 0.19, r_c can have nonzero value

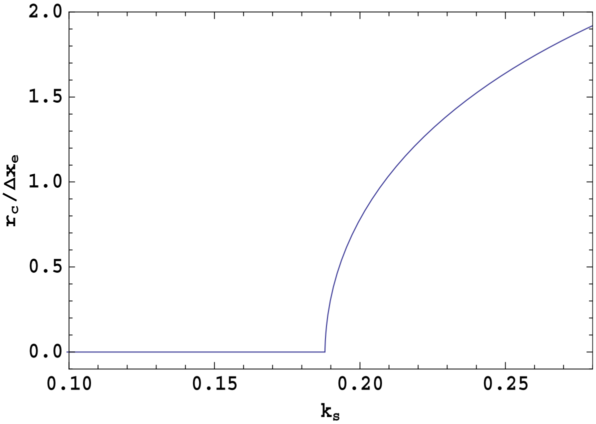
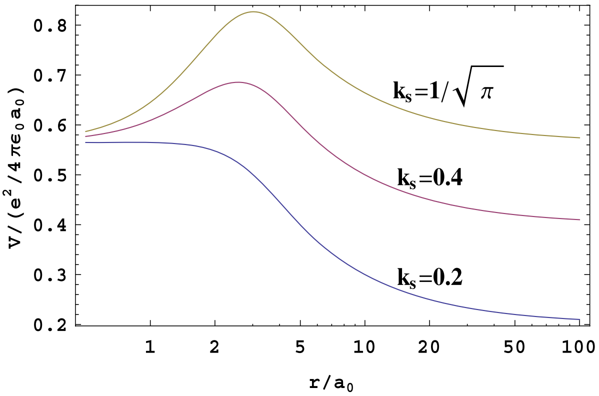


Fig. 2 Configuration of the potential, last two terms of (46), for $k_s = 0.2,\ 0.4,\ 1/\sqrt{\pi}$. The potential of $r = 0$ is $V = (1/\sqrt{\pi})(e^2/4\pi\epsilon_0 a_0)$ while $r = \infty$ leads $V = k_s(e^2/4\pi\epsilon_0 a_0)$. If $k_s > 1/\sqrt{\pi}$, we obtain $V(r = 0) < V(r = \infty)$



For the equation $\partial E/\partial r = 0$, nonzero solution about r that we write r_c can be found when the first term is larger than some critical value. It can be characterized by the next parameter

$$k_s = \frac{3\alpha m_e D_e^2 e^{-\frac{(\Delta x_e)^2(\boldsymbol{p}_1-\boldsymbol{p}_2)^2}{\hbar^2}}}{\hbar c \Delta x_e}. \tag{48}$$

If k_s is roughly larger than 0.19, r_c can have nonzero value as described in Fig. 1. In that case, the two electrons will be stable at $r = 0$, see Fig. 2. Therefore the condition to make a Cooper-pair can read

$$\frac{3\alpha m_e D_e^2 e^{-\frac{(\Delta x_e)^2(\boldsymbol{p}_1-\boldsymbol{p}_2)^2}{\hbar^2}}}{\hbar c \Delta x_e} \gtrsim 0.19. \tag{49}$$

This can be regarded as the condition for the diffusion coefficient if the difference of momentums is negligible and the amount of position fluctuation is known. For instance, if $\boldsymbol{p}_1 \sim \boldsymbol{p}_2$ and $\Delta x_e \sim a_0$, we have $D_e \gtrsim 0.01$ [m^2/s].

Next, we consider the gap equation. Defining the energy gap, from infinite distance to combined state, by $V_{\text{gap}} = V(r=0) - V(r=\infty)$ with the constraint $\boldsymbol{p}_1 = \boldsymbol{p}_2$, we have

$$V_{\text{gap}} = \frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{\pi}} - k_s\right). \tag{50}$$

Then, it can be negative when $k_s > 1/\sqrt{\pi}$. Rewriting it to the relation with the diffusion coefficient by assuming $\Delta x_e \sim a_0$, we have $D > 1.7 \times 10^{-2}$ [m^2/s]. At a rough estimation, we can approximately obtain usual gap equation. That is, considering that electrons on the Fermi surface with a width of $\hbar\omega_c$ form Cooper pairs, we can estimate

$$E \approx 2E_F - 2\hbar\omega_c e^{2/N(0)V_{\text{gap}}}. \tag{51}$$

To make more correct prediction, it needs to estimate the gap equation with our potential including Gaussian and error functions. Numerical result with the Gaussian potential is obtained by [29]. Our model seems applicable in the same method.

It should be noted that the approximate solution of two Gaussian waves are valid only if they are well separated, then the value of $V(r=0)$ is not reliable. Besides, the stable point might be altered to $r \neq 0$. Nonlinear analysis will be necessary to make it precise.

7 Conclusion

We have analyzed a new interaction term that is led from the Hamiltonian of the Nelson equations with a bit of modification. It is interesting because the interaction does not have a coupling constant of free parameter. By measuring the position fluctuation and the diffusion coefficient, the interaction can be evaluated. As it is attractive between spin up and spin down, the stochastic interaction can exceed the Coulomb repulsion. Our prediction is the formation of Cooper-pairs and positive binding energy relating to a specific value of a new parameter k_s. With this parameter, the critical temperature can be obtained with assuming phenomenological expression of the diffusion coefficient.

There can be wide applications of our quantization scheme of the Brownian motion. For example, nuclear and atom will have complicated structures due to nonlinearity when the stochastic potential is applied. Including the shift of energy spectrum of hydrogen and helium, it modifies conventional results of quantum mechanics. In other words, investigating any quantum effect with non-negligible diffusion coefficient, there must be some anomalous correction. Searching the dependence of the position fluctuation and the diffusion coefficient for this, it seems easy to confirm the model.

The strongest prediction is that the critical temperature of superconductor will be influenced by the diffusion coefficient with a specific function. Since the attraction between two electrons can be determined by measurable parameters, it will predict several other things in phenomena of superconductivity. Seeking the relation between superconductivity and diffusion, the configuration of the stochastic potential will be probed via the attraction of Cooper-pairs. It can be a great advantage if superconductivity is resolved by a few known parameters.

References

1. Hu, B.L., Matacz, A.: Phys. Rev. D **49**, 6612 (1994)
2. Hänggi, P., Ingold, G.-L.: Chaos **15**, 026105 (2005)

3. Rabei, E.M., Ajlouni, A.-W., Ghassib, H.: Int. J. Theor. Phys. **45**, 1619 (2006)
4. Gour, G., Sriramkumar, L.: Found. Phys. **29**, 1917 (1999)
5. Jaekel, M.-T.: Rep. Prog. Phys. **60**, 863 (1997)
6. Ayik, S., Washiyama, K., Lacroix, D.: Phys. Rev. C **79**, 054606 (2009)
7. Ankerhold, J., Grabert, H., Pechukas, P.: Chaos **15**, 026106 (2005)
8. Nelson, E.: Phys. Rev. **150**, 1079 (1966)
9. Namiki, M.: Prog. Theor. Phys. **111**, 1 (1993)
10. de la Peña, L., Cetto, A.M.: Found. Phys. **12**, 1017 (1982)
11. Parisi, G., Wu, Y.: Sci. Sin. **24**, 483 (1981)
12. Hüffel, H., Kelnhofer, G.: Phys. Lett. B **588**, 145 (2004)
13. Hüffel, H., Kelnhofer, G.: Ann. Phys. **266**, 417 (1998)
14. Marculescu, S., Okano, K., Schülke, L., Zheng, B.: Nucl. Phys. B **417**, 579 (1994)
15. Menezes, G., Svaiter, N.F.: Physica A **374**, 617 (2007)
16. Harvey, R.J.: Phys. Rev. **152**, 1115 (1966)
17. Cresson, J., Darses, S.: J. Math. Phys. **48**, 072703 (2007)
18. Grössing, G.: Found. Phys. Lett. **17**, 343 (2004)
19. Suzuki, Y., Koshizuka, S., Oka, Y.: Comput. Methods Appl. Mech. Eng. **196**, 2876 (2007)
20. Anthony, K.-H.: J. Non-Newton. Fluid Mech. **96**, 291 (2001)
21. Morrison, P.J.: Rev. Mod. Phys. **70**, 467 (1998)
22. Pushkarov, D.I.: Cent. Eur. J. Phys. **2**, 420 (2004)
23. Janiš, V., Kolorenč, J., Špička, V.: Eur. J. Phys. B **35**, 77 (2003)
24. Belavkin, V.P.: Theory Probab. Appl. **38**, 742 (1993)
25. Shalizi, C.: 36-754, Advanced Probability II or Almost None of the Theory of Stochastic Processes. http://www.stat.cmu.edu/~cshalizi/754/notes/
26. Davidson, M.: Lett. Math. Phys. **3**, 271 (1979)
27. Grössing, G.: Phys. Lett. A **296**, 1 (2002)
28. Pavon, M.: J. Math. Phys. **42**, 4846 (2001)
29. Milstein, J.N., Burnett, K.: J. Phys. B: At. Mol. Opt. Phys. **39**, 1965 (2006)